

A new approach of the Chebyshev wavelets for the variable-order time fractional mobile-immobile advection-dispersion model

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Abstract

This paper proposes a new numerical method based on the Chebyshev wavelets (CWs) to solve the variable-order time fractional mobile-immobile advection-dispersion equation. To do this, a new operational matrix of variable-order fractional derivative in the Caputo sense for the CWs is derived and is used to obtain an approximate solution for the problem under study. Along the way a new family of piecewise functions is introduced and employed to derive a general method to compute this matrix. The main advantage behind the proposed approach is that the problem under consideration is transformed into a linear system of algebraic equations. So, it can be solved simply to obtain an approximate solution. The efficiency and accuracy of the proposed method are shown for some concrete examples. These results show that the proposed method is very efficient and accurate.

Keywords: Chebyshev wavelets (CWs); Operational matrix of variable-order fractional derivative; Variable-order time fractional mobile-immobile advection-dispersion equation; Caputo's variable-order fractional derivative.

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1 Introduction

Variable-order fractional derivatives, which are an extension of constant-order fractional ones have been introduced in several physical applications [1–3]. Recently, some researchers [4–11] have shown that many complex physical models can be described via variable-order derivatives with a great success. It is worth noting that analytically handling equations described by the variable-order fractional derivatives is difficult due to their highly complex, so proposing efficient methods to find their numerical solutions is of great importance in practical. So, recently several methods have been proposed to solve variable-order fractional differential equations numerically such as [12–27].

Wavelets theory which is a relatively new area in mathematical research has been applied in a wide range of engineering disciplines [28]. In recent years, wavelets have been applied for solving different types of partial differential equations e.g. [28–31].

The aim of this paper is to propose a new numerical method based on the CWs to solve the following variable-order time fractional mobile-immobile advection-dispersion model [32]:

$$\alpha_1 \frac{\partial u(x, t)}{\partial t} + \alpha_2 {}^c D_t^{\gamma(x, t)} u(x, t) = -\mu_1 \frac{\partial u(x, t)}{\partial x} + \mu_2 \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t), \quad (x, t) \in \Omega, \quad (1.1)$$

with $\Omega = [0, 1] \times [0, 1]$, subject to the following initial-boundary conditions:

$$\begin{aligned} u(x, 0) &= g(x), \\ u(0, t) &= h_1(t), \quad u(1, t) = h_2(t), \end{aligned} \quad (1.2)$$

where $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\mu_1 > 0$, $\mu_2 > 0$, $0 < \gamma(x, t) \leq 1$, $f(x, t)$, $g(x)$, $h_1(t)$ and $h_2(t)$ are given functions, and ${}_0^c D_t^{\gamma(x, t)}$ denotes the variable-order fractional derivatives in the Caputo sense of order $0 < \gamma(x, t) \leq 1$, as defined by [19, 20]:

$${}_0^c D_t^{\gamma(x, t)} u(x, t) = \frac{1}{\Gamma(1 - \gamma(x, t))} \int_0^t (t - s)^{-\gamma(x, t)} \frac{\partial u(x, s)}{\partial s} ds, \quad t > 0. \quad (1.3)$$

It is worth noting that based on the definition of the variable-order fractional derivative in the Caputo sense as [20], we have the following useful property:

$${}_0^c D_t^{\vartheta(x, t)} t^m = \begin{cases} \frac{\Gamma(m + 1)}{\Gamma(m - \vartheta(x, t) + 1)} t^{m - \vartheta(x, t)}, & q \leq m \in \mathbb{N}, \\ 0, & o.w, \end{cases} \quad (1.4)$$

where $q - 1 < \vartheta(x, t) \leq q$.

To solve equation in (1.1), we first derive a new operational matrix of variable-order fractional derivative in the Caputo sense for the CWs and then, employ this matrix to obtain an approximate solution for the problem at hand. Along the way, a new family of piecewise functions is introduced and employed to derive a general procedure for forming this matrix.

In the proposed method, at first the solution of the problem at hand is expanded in terms of the CWs. Then, by computing the operational matrix of variable-order fractional derivative and using some properties of these basis polynomials, we transform its solution to the solution of a linear system of algebraic equations. This greatly simplifies the process of solving the problem as well as help to achieve an approximate solution for the problem.

The remainder of this paper is organized as follows: In section 2, the CWs and their properties are introduced. In section 3, the operational matrix of variable-order fractional derivative for the CWs is derived and in section 4, the proposed method is described for solving the problem under study. Section 5, contains some numerical examples which are solved using the proposed method. Finally, a conclusion is given in section 6.

2 The CWs and their properties

Wavelets constitute a family of functions constructed from dilation and translation of a single function $\psi(t)$ called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets as:

$$\psi_{ab}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t - b}{a}\right), \quad a, b \in \mathbb{R}, \quad a \neq 0. \quad (2.1)$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, where $a_0 > 1$, $b_0 > 0$, we have the following family of discrete wavelets

$$\psi_{kn}(t) = |a_0|^{\frac{k}{2}} \psi(a_0^k t - nb_0), \quad k, n \in \mathbb{Z}, \quad (2.2)$$

where the functions $\psi_{kn}(t)$ form a wavelet basis for $L^2(\mathbb{R})$.

In practice, when $a_0 = 2$ and $b_0 = 1$, the functions $\psi_{kn}(t)$ form an orthonormal basis.

The CWs are defined on the interval $[0, 1]$ by [33, 34]:

$$\psi_{nm}(t) = \begin{cases} \beta_m 2^{\frac{k}{2}} T_m^*(2^k t - n), & t \in \left[\frac{n}{2^k}, \frac{n+1}{2^k}\right], \\ 0, & o.w, \end{cases} \quad (2.3)$$

for $n = 0, 1, \dots, 2^k - 1$, $m = 0, 1, \dots, M - 1$, $(k, M) \in \mathbb{N}$, where

$$\beta_m = \begin{cases} \sqrt{\frac{2}{\pi}}, & m = 0, \\ \frac{2}{\sqrt{\pi}}, & m \geq 1, \end{cases} \quad (2.4)$$

and $T_m^*(t)$ denotes the shifted Chebyshev polynomials, which are defined on the interval $[0, 1]$ as:

$$\begin{aligned} T_0^*(t) &= 1, \\ T_m^*(t) &= m \sum_{i=0}^m (-1)^{m-i} \frac{2^{2i} (m+i-1)!}{(m-i)!(2i)!} t^i, \quad m = 1, 2, \dots \end{aligned} \quad (2.5)$$

The set of the CWs is an orthogonal set with respect to the weight function $w_n(t)$ where

$$w_n(t) = \begin{cases} \frac{1}{\sqrt{1 - (2^{k+1}t - 2n - 1)^2}}, & t \in \left[\frac{n}{2^k}, \frac{n+1}{2^k} \right], \\ 0, & o.w. \end{cases} \quad (2.6)$$

The CWs can be used to expand any function $u(t)$ which is defined over $[0, 1]$ as:

$$u(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(t), \quad (2.7)$$

where $c_{nm} = \langle u(t), \psi_{nm}(t) \rangle$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $L_{w_n}^2[0, 1]$. By truncating the infinite series in equation (2.7), $u(t)$ is approximated as:

$$u(t) \simeq \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(t) = C^T \Psi(t), \quad (2.8)$$

where C and $\Psi(t)$ are column vectors with $\hat{m} = 2^k M$ elements. For simplicity, equation (2.8) is written as:

$$u(t) \simeq \sum_{i=1}^{\hat{m}} c_i \psi_i(t) = C^T \Psi(t), \quad (2.9)$$

where $c_i = c_{nm}$ and $\psi_i(t) = \psi_{nm}(t)$, and the index i is calculated as $i = Mn + m + 1$. Thus, we have:

$$C \triangleq [c_1, c_2, \dots, c_{\hat{m}}]^T,$$

$$\Psi(t) \triangleq [\psi_1(t), \psi_2(t), \dots, \psi_{\hat{m}}(t)]^T. \quad (2.10)$$

Similarly, the CWs can be used to expand an arbitrary function of two variables such as $u(x, t)$ which is defined over $[0, 1] \times [0, 1]$ as:

$$u(x, t) \simeq \sum_{i=1}^{\hat{m}} \sum_{j=1}^{\hat{m}} u_{ij} \psi_i(x) \psi_j(t) = \Psi^T(x) \mathbf{U} \Psi(t), \quad (2.11)$$

where $U = [u_{ij}]$ and $u_{ij} = (\psi_i(x), (u(x, t), \psi_j(t)))$.

The derivative of the vector $\Psi(t)$ defined in equation (2.10) can be expressed as [35]:

$$\frac{d\Psi(t)}{dt} = \mathbf{D}\Psi(t), \quad (2.12)$$

where \mathbf{D} is the $\hat{m} \times \hat{m}$ operational matrix of one-time derivative of the CWs vector and is given by:

$$\mathbf{D} = \begin{pmatrix} \mathbf{F} & 0 & \dots & 0 \\ 0 & \mathbf{F} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \mathbf{F} \end{pmatrix}, \quad (2.13)$$

where \mathbf{F} is an $M \times M$ matrix with the elements:

$$\mathbf{F}_{ij} = \begin{cases} 2^{k+2}(i-1)\sqrt{\frac{\sigma_{i-1}}{\sigma_{j-1}}}, & i = 2, 3, \dots, M, \quad j = 1, 2, \dots, i-1, \quad i+j \text{ is odd,} \\ 0, & o.w, \end{cases} \quad (2.14)$$

and

$$\sigma_j = \begin{cases} 2, & j = 0, \\ 1, & j \geq 1. \end{cases}$$

In general, the operational matrix \mathbf{D}^r of r -times derivative of $\Psi(t)$ can be expressed as:

$$\frac{d^r \Psi(t)}{dt^r} = \mathbf{D}^r \Psi(t), \quad (2.15)$$

where \mathbf{D}^r is the r -th power of matrix \mathbf{D} .

3 The operational matrix of variable-order fractional derivative

The variable-order fractional derivative of order $(q-1) < \vartheta(x, t) \leq q$, $q \in \mathbb{N}$, of the vector $\Psi(t)$ which is defined in equation (2.10) can be expressed as:

$${}_0^c D_t^{\vartheta(x, t)} \Psi(t) \simeq \mathbf{Q}_t^{\vartheta(x, t)} \Psi(t), \quad (3.1)$$

where $\mathbf{Q}_t^{\vartheta(x, t)}$ is called the $\hat{m} \times \hat{m}$ operational matrix of variable-order fractional derivative of order $\vartheta(x, t)$ for the CWs.

In the sequel, we will derive an explicit form for this matrix. To this end, we introduce another family of piecewise functions, which are defined on $[0, 1]$ as:

$$\phi_{nm}(t) = \begin{cases} t^m, & t \in \left[\frac{n}{2^k}, \frac{n+1}{2^k} \right], \\ 0, & o.w, \end{cases} \quad (3.2)$$

for $n = 0, 1, \dots, 2^k - 1$, $m = 0, 1, \dots, M-1$.

Unlike the CWs, this family of functions is not normalized. An \hat{m} -set of these functions may be expressed as:

$$\Phi(t) \triangleq [\phi_1(t), \phi_2(t), \dots, \phi_{\hat{m}}(t)]^T, \quad (3.3)$$

where $\phi_i(t) = \phi_{nm}(t)$, and the index i is determined by the relation $i = Mn + m + 1$.

The following relation holds among these functions and the CWs:

$$\Phi(t) = \mathbf{P}\Psi(t), \quad (3.4)$$

where $p_{ij} = (\phi_i, \psi_j)$.

Lemma 3.1. Let $\phi_{nm}(t)$ be as defined in equation (3.2), and $(q-1) < \vartheta(x, t) \leq q$ be a positive function defined over $[0, 1]$. Then, we have:

$${}_0^c D_t^{\vartheta(x, t)} \phi_{nm}(t) = \begin{cases} \frac{m!}{\Gamma(m - \vartheta(x, t) + 1)} t^{m - \vartheta(x, t)}, & m = q, q+1, \dots, M-1, \quad t \in \left[\frac{n}{2^k}, \frac{n+1}{2^k} \right], \\ 0, & o.w. \end{cases}$$

Proof. By considering relation (1.4), the proof will be straightforward. \square

Theorem 3.2. Let $\Phi(t)$ be the piecewise functions vector defined as in equation (3.2) and $(q-1) < \vartheta(x, t) \leq q$ be a positive function defined over $[0, 1]$. The variable-order fractional derivative of order $\vartheta(x, t)$ in the Caputo sense of $\Phi(t)$ can be expressed by:

$${}_0^c D_t^{\vartheta(x, t)} \Phi(t) = \mathbf{T}_t^{\vartheta(x, t)} \Phi(t),$$

where $\mathbf{T}_t^{\vartheta(x, t)}$ is an $\hat{m} \times \hat{m}$ matrix given by:

$$\mathbf{T}_t^{\vartheta(x, t)} = \begin{pmatrix} \mathbf{G}_t^{\vartheta(x, t)} & 0 & 0 & \dots & 0 \\ 0 & \mathbf{G}_t^{\vartheta(x, t)} & 0 & \dots & 0 \\ 0 & 0 & \mathbf{G}_t^{\vartheta(x, t)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \mathbf{G}_t^{\vartheta(x, t)} \end{pmatrix}, \quad (3.5)$$

and $\mathbf{G}_t^{\vartheta(x, t)}$ is an $M \times M$ matrix given as:

$$\mathbf{G}_t^{\vartheta(x, t)} = t^{-\vartheta(x, t)} \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dots & 0 & \frac{q!}{\Gamma(q - \vartheta(x, t) + 1)} & 0 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 & \frac{(q+1)!}{\Gamma(q - \vartheta(x, t) + 2)} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \frac{(M-2)!}{\Gamma(M - \vartheta(x, t) - 1)} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \frac{(M-1)!}{\Gamma(M - \vartheta(x, t))} \end{pmatrix}.$$

Proof. It is an immediate consequence of Lemma 3.1. \square

To illustrate the calculation procedure, we choose $(M = 5, k = 1)$ and $1 < \vartheta(x, t) \leq 2$. Thus, we have:

$$\mathbf{G}_t^{\vartheta(x, t)} = t^{-\vartheta(x, t)} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\Gamma(3 - \vartheta(x, t))} & 0 & 0 \\ 0 & 0 & 0 & \frac{6}{\Gamma(4 - \vartheta(x, t))} & 0 \\ 0 & 0 & 0 & 0 & \frac{24}{\Gamma(5 - \vartheta(x, t))} \end{pmatrix}.$$

Theorem 3.3. Let $\Psi(t)$ be the CWs vector defined in equation (2.10) and $(q-1) < \vartheta(x, t) \leq q$ be a positive function defined over $[0, 1]$. The variable-order fractional derivative of order $\vartheta(x, t)$ in the Caputo sense of $\Psi(t)$ can be expressed as:

$${}_0^c D_t^{\vartheta(x, t)} \Psi(t) = \mathbf{Q}_t^{\vartheta(x, t)} \Psi(t) = \left(\mathbf{P}^{-1} \mathbf{T}_t^{\vartheta(x, t)} \mathbf{P} \right) \Psi(t), \quad (3.6)$$

where \mathbf{P} is the coefficients matrix defined in equation (3.4), $\mathbf{T}_t^{\vartheta(x, t)}$ is the operational matrix of variable-order fractional derivative of order $\vartheta(x, t)$ for the piecewise functions, which is defined in equation (3.5) and $\mathbf{Q}_t^{\vartheta(x, t)}$ is called the operational matrix of variable-order fractional derivative of order $\vartheta(x, t)$ for the CWs.

Proof. By considering equation (3.4) and Theorem 3.2, we have:

$$\Psi(t) = \mathbf{P}^{-1} \Phi(t),$$

and then

$${}_0^c D_t^{\vartheta(x, t)} \Psi(t) = \mathbf{P}^{-1} {}_0^c D_t^{\vartheta(x, t)} \Phi(t) = \mathbf{P}^{-1} \mathbf{T}_t^{\vartheta(x, t)} \Phi(t) = \left(\mathbf{P}^{-1} \mathbf{T}_t^{\vartheta(x, t)} \mathbf{P} \right) \Psi(t),$$

which completes the proof. \square

To illustrate the calculation procedure, we choose $(M = 3, k = 1)$ and $0 < \vartheta(x, t) \leq 1$. Thus, we have:

$$\mathbf{Q}_t^{\vartheta(x, t)} = t^{-\vartheta(x, t)} \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \end{pmatrix},$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\sqrt{2}}{\Gamma(2 - \vartheta(x, t))} & \frac{1}{\Gamma(2 - \vartheta(x, t))} & 0 \\ \frac{-4\sqrt{2}}{\Gamma(2 - \vartheta(x, t))} + \frac{6\sqrt{2}}{\Gamma(3 - \vartheta(x, t))} & \frac{-4}{\Gamma(2 - \vartheta(x, t))} + \frac{8}{\Gamma(3 - \vartheta(x, t))} & \frac{2}{\Gamma(3 - \vartheta(x, t))} \end{pmatrix},$$

and

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{3\sqrt{2}}{\Gamma(2 - \vartheta(x, t))} & \frac{1}{\Gamma(2 - \vartheta(x, t))} & 0 \\ \frac{-36\sqrt{2}}{\Gamma(2 - \vartheta(x, t))} + \frac{38\sqrt{2}}{\Gamma(3 - \vartheta(x, t))} & \frac{-12}{\Gamma(2 - \vartheta(x, t))} + \frac{24}{\Gamma(3 - \vartheta(x, t))} & \frac{2}{\Gamma(3 - \vartheta(x, t))} \end{pmatrix}.$$

4 Description of the proposed method

In this section, the CWs expansion and their operational matrix of variable-order fractional derivative are used together to solve the variable-order time fractional mobile-immobile advection-dispersion model of equation (1.1). To solve this equation, we approximate the unknown function $u(x, t)$ by the CWs as:

$$u(x, t) \simeq \Psi(x)^T \mathbf{U} \Psi(t), \quad (4.1)$$

where $\mathbf{U} = [u_{ij}]$ is an $\hat{m} \times \hat{m}$ matrix which we need to compute it, and $\Psi(\cdot)$ is the CWs vector, which is defined in equation (2.10).

By derivatives of equation (4.1) for one time with respect to t and two times with respect to x , and using equations (2.12) and (2.15), we obtain:

$$u_t(x, t) \simeq \Psi(x)^T \mathbf{U} \mathbf{D} \Psi(t), \quad u_x(x, t) \simeq \Psi(x)^T \mathbf{D}^T \mathbf{U} \Psi(t), \quad u_{xx}(x, t) \simeq \Psi(x)^T (\mathbf{D}^2)^T \mathbf{U} \Psi(t). \quad (4.2)$$

By the variable-order fractional derivative of order $\gamma(x, t)$ of equation (4.1) with respect to t , and considering equation (3.6), we have:

$${}_0^c D_t^{\gamma(x, t)} u(x, t) \simeq \Psi(x)^T \mathbf{U} \mathbf{Q}^{\gamma(x, t)} \Psi(t). \quad (4.3)$$

Applying equation (4.1) into the initial-boundary conditions expressed in equation (1.2), and using equation (2.12), we have:

$$\Lambda_1(x) = \Psi(x)^T \mathbf{U} \Psi(0) - g(x) \simeq 0, \quad \Lambda_2(t) = \Psi(0)^T \mathbf{U} \Psi(t) - h_1(t) \simeq 0, \quad \Lambda_3(t) = \Psi(1)^T \mathbf{U} \Psi(t) - h_2(t) \simeq 0. \quad (4.4)$$

By substituting equations (4.2) and (4.3) into the variable-order time fractional mobile-immobile advection-dispersion model in equation (1.1), we get:

$$\Psi(x)^T \left(\alpha_1 \mathbf{U} \mathbf{D} + \alpha_2 \mathbf{U} \mathbf{Q}^{\alpha(x, t)} + \mu_1 \mathbf{D}^T \mathbf{U} - \mu_2 (\mathbf{D}^2)^T \mathbf{U} \right) \Psi(t) - f(x, t) \triangleq F(x, t) \simeq 0. \quad (4.5)$$

In order to obtain an approximate solution for the problem at hand, we need to find the unknown matrix \mathbf{U} . So, we need to construct a linear system of \hat{m}^2 equations which by solving it, the unknown matrix \mathbf{U} is determined. To this end, we choose $\hat{m}^2 - 3\hat{m} + 2$ algebraic equations using equation (4.5) as:

$$F(x_i, t_j) = 0, \quad i = 2, 3, \dots, \hat{m} - 1, \quad j = 2, 3, \dots, \hat{m}, \quad (4.6)$$

where x_i and t_j are the zeros of the shifted Chebyshev polynomials of degree \hat{m} on $[0, 1]$.

Moreover, by taking the collocation points x_i and t_i into equation (4.4) as:

$$\begin{aligned} \Lambda_1(x_i) &= 0, \quad i = 1, 2, \dots, \hat{m}, \\ \Lambda_2(t_i) &= 0, \quad i = 2, 3, \dots, \hat{m}, \\ \Lambda_3(t_i) &= 0, \quad i = 2, 3, \dots, \hat{m}, \end{aligned} \quad (4.7)$$

we get $3\hat{m} - 2$ linear algebraic equations.

Combining equations (4.6) and (4.7) gives a linear system of \hat{m}^2 algebraic equations, which can be solved for the unknown matrix $\mathbf{U} = [u_{ij}]$, $i, j = 1, 2, \dots, \hat{m}$, using MAPLE or MATLAB software packages. By determining \mathbf{U} , we can determine the approximate solutions for $u(x, t)$ from equation (4.1).

5 Illustrative test problems

In this section, we provide some numerical examples to demonstrate the efficiency and reliability of our method. It is worth mentioning that all the numeric computations are performed by MAPLE 15 with 50 decimal digits.

Example 1. Consider the variable-order time fractional problem [32]:

$$\alpha_1 \frac{\partial u(x, t)}{\partial t} + \alpha_2 {}_0^c D_t^{\gamma(x, t)} u(x, t) = -\mu_1 \frac{\partial u(x, t)}{\partial x} + \mu_2 \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t),$$

subject to the following initial-boundary conditions:

$$u(x, 0) = 10x^2(1 - x)^2, \quad u(0, t) = 0, \quad u(1, t) = 0,$$

where

$$f(x, t) = 10 \left(\alpha_1 + \alpha_2 \frac{t^{1-\gamma(x, t)}}{\Gamma(2-\gamma(x, t))} \right) x^2 (1-x)^2 + 10 (\mu_1 (4x^3 - 6x^2 + 2x) - \mu_2 (12x^2 - 12x + 2)) (t+1).$$

The analytical solution for this problem is $u(x, t) = 10(t+1)x^2(1-x)^2$. The numerical solution for this problem is also computed by our method for $\gamma(x, t) = 1 - 0.5e^{-(xt)}$, $\alpha_1 = \alpha_2 = \mu_1 = \mu_2 = 1$ and $(k = 1, M = 5)$. The numerical behavior of the approximate solution and absolute error are shown in Fig. 1. From Fig. 1 it can be seen that the proposed method is very efficient and accurate for solution of this problem. It is also worth noting that in [29], the authors have proposed a discrete implicit numerical method for solving this problem. By considering Fig. 1 and Tables 1 and 2 in [29], one can simply see that the approximate solution obtained by the method of this paper is more accurate the one in [29]. Moreover, the implementation of our proposed method is much simple in comparison with the one in [29].

Example 2. Consider the following variable-order time fractional problem:

$$\alpha_1 \frac{\partial u(x, t)}{\partial t} + \alpha_2 {}^c D_t^{\gamma(x, t)} u(x, t) = -\mu_1 \frac{\partial u(x, t)}{\partial x} + \mu_2 \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t),$$

with homogenous initial-boundary conditions and

$$f(x, t) = 10 \left(\alpha_1 (3t^2 - 4t^3) + \alpha_2 \left(\frac{6t^{3-\gamma(x, t)}}{\Gamma(4-\gamma(x, t))} - \frac{24t^{4-\gamma(x, t)}}{\Gamma(5-\gamma(x, t))} \right) \right) x^3 (1-x) \\ + 10 (\mu_1 (3x^2 - 4x^3) - \mu_2 (6x - 12x^2)) t^3 (1-t).$$

Its analytical solution is $u(x, t) = 10x^3t^3(1-x)(1-t)$. It is also solved numerically by our method for $\gamma(x, t) = 1 - 0.4 \sin(x+t)^2$, $\alpha_1 = \alpha_2 = 1$, $\mu_1 = \mu_2 = 2$ and $(k = 1, M = 5)$. The numerical behavior of the approximate solution and absolute error are shown in Fig. 2. From Fig. 2, it can be seen that our method is very efficient and accurate for solving this problem.

Example 3. Consider the following variable-order time fractional problem:

$$\alpha_1 \frac{\partial u(x, t)}{\partial t} + \alpha_2 {}^c D_t^{\gamma(x, t)} u(x, t) = -\mu_1 \frac{\partial u(x, t)}{\partial x} + \mu_2 \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t),$$

subject to the following initial-boundary conditions:

$$u(x, 0) = 0, \quad u(0, t) = t^3, \quad u(1, t) = e t^3,$$

where

$$f(x, t) = \left(\alpha_1 3t^2 + \alpha_2 \frac{6t^{3-\gamma(x, t)}}{\Gamma(4-\gamma(x, t))} + (\mu_1 - \mu_2) t^3 \right) x (1-x) e^x.$$

The analytical solution for this problem is $u(x, t) = t^3 e^x$. It is also solved numerically by our method for $\gamma(x, t) = 0.8 + 0.2e^{-x} \sin(t)$, $\alpha_1 = 1$, $\alpha_2 = \frac{1}{2}$, $\mu_1 = 1$, $\mu_2 = 2$, $k = 0$, and some different values of M . The absolute errors of the approximate solution at $x = 0.5$ for some different values of t are shown in Table 1. The numerical behavior of the approximate solution and absolute error for $M = 11$ are shown in Fig. 3. From Table 1, we observe that the proposed method can provide numerical results with high accuracy in all cases. Furthermore, it can be seen that the accuracy of the obtained results is improved by increasing the number of the CWs. From Fig. 3, it can be seen that our method is very efficient and accurate for solution of this problem.

Example 4. Consider the following variable-order time fractional problem:

$$\frac{\partial u(x, t)}{\partial t} + {}^c D_t^{\gamma(x, t)} u(x, t) = -\frac{\partial u(x, t)}{\partial x} + \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t),$$

subject to the following initial-boundary conditions:

$$u(x, 0) = 0, \quad u(0, t) = t^3, \quad u(1, t) = t^3,$$

where

$$f(x, t) = \left(3t^2 + 6 \frac{t^{3-\alpha(x,t)}}{\Gamma(4-\alpha(x,t))} \right) (|2x-1|)^3 + (6|2x-1|(2x-1) - 24|2x-1|) t^3.$$

Its analytical solution is $u(x, t) = t^3 (|2x-1|)^3$. It is also solved numerically by the our method for $\gamma(x, t) = 1 - e^{-xt}$ and $(k = 1, M = 4)$. The numerical behavior of the approximate solution and absolute error are shown in Fig. 4. By Fig. 4, it can be seen that our method is very efficient and accurate for solving this problem.

Table 1: The absolute errors of the approximate solution at $x = 0.5$ for different values of t for Example 3.

t	$M = 4$	$M = 5$	$M = 6$	$M = 7$	$M = 8$	$M = 9$	$M = 10$	$M = 11$
0.1	7.596E-07	1.492E-09	1.785E-09	2.336E-12	6.160E-13	5.243E-16	1.089E-15	9.735E-19
0.2	9.642E-08	1.492E-09	1.358E-08	2.325E-11	2.099E-12	1.240E-14	1.114E-14	1.336E-17
0.3	4.509E-07	2.284E-08	5.224E-08	9.903E-11	8.307E-13	6.190E-14	4.314E-14	5.765E-17
0.4	5.420E-06	6.907E-08	1.350E-07	2.687E-10	1.491E-11	1.798E-13	1.114E-13	1.574E-16
0.5	1.783E-05	1.571E-07	2.796E-07	5.732E-10	4.758E-11	3.989E-13	2.306E-13	3.372E-16
0.6	4.070E-05	3.010E-07	5.041E-07	1.054E-09	1.064E-10	7.525E-13	4.157E-13	6.219E-16
0.7	7.706E-05	5.150E-07	8.264E-07	1.754E-09	1.994E-10	1.274E-12	6.816E-13	1.036E-15
0.8	1.299E-04	8.138E-07	1.265E-06	2.717E-09	3.342E-10	1.998E-12	1.043E-12	1.607E-15
0.9	2.022E-04	1.212E-06	1.838E-06	3.984E-09	5.188E-10	2.959E-12	1.515E-12	2.359E-15

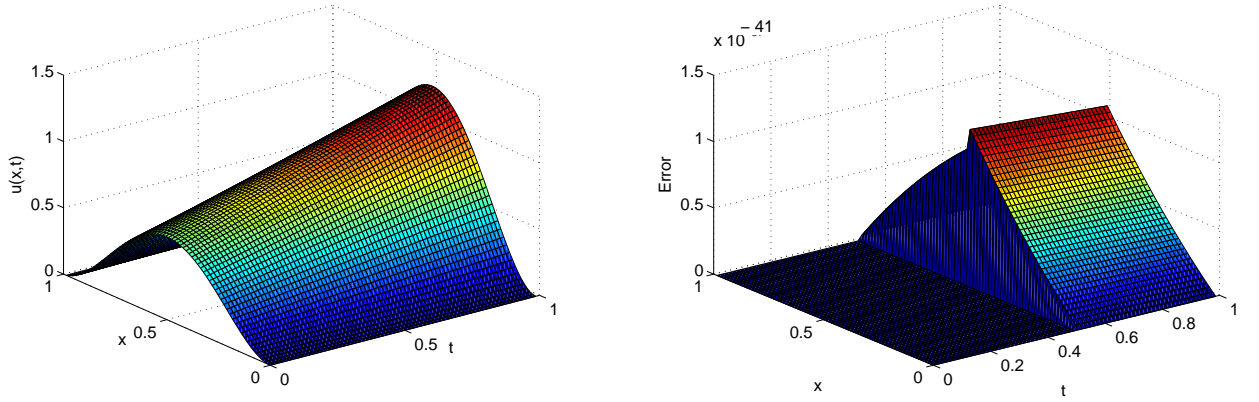


Fig. 1: The numerical behavior of the approximate solution (left) and absolute error (right) for Example 1.

6 Conclusion

In this paper, a new numerical method based on the CWs was proposed to obtain an approximate solution for the variable-order time fractional mobile-immobile advection-dispersion model. To this end, a new operational matrix of variable-order fractional derivative for the CWs was obtained and employed to obtain the approximate solution for the problem under study. Along the way a new family of piecewise functions was introduced

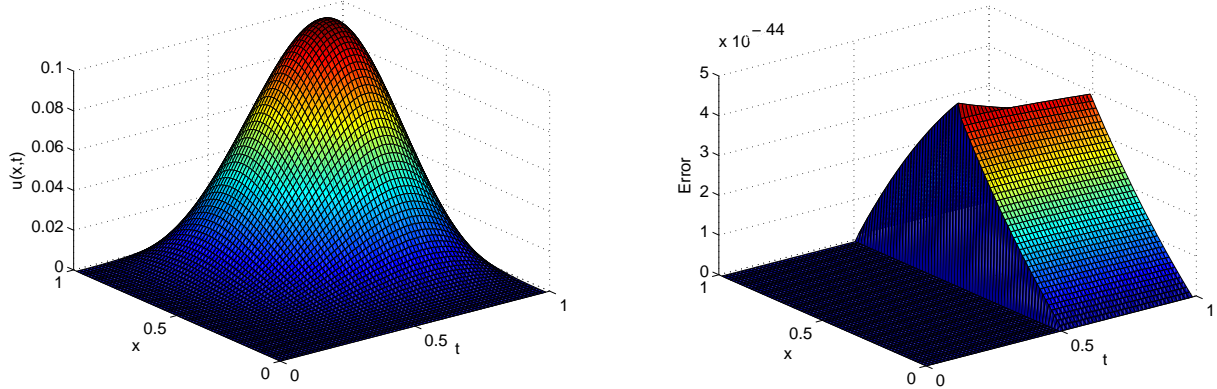


Fig. 2: The numerical behavior of the approximate solution (left) and absolute error (right) for Example 2.

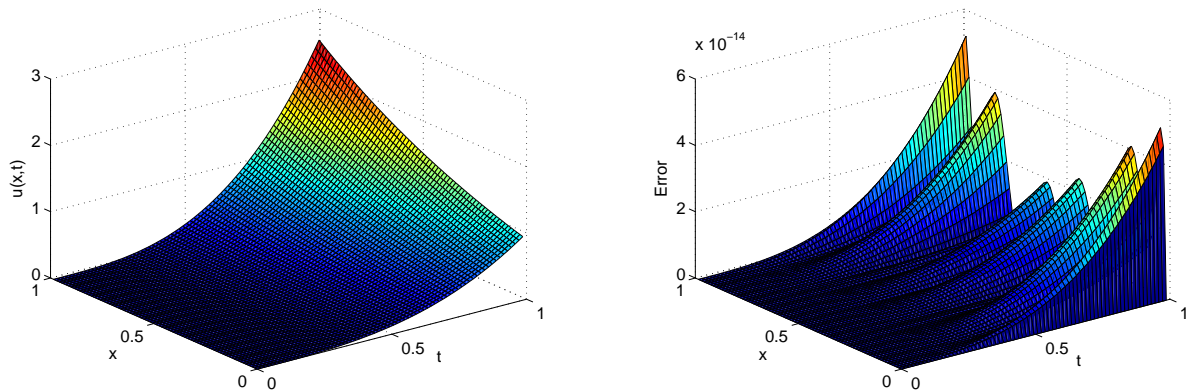


Fig. 3: The numerical behavior of the approximate solution (left) and absolute error (right) for Example 3.

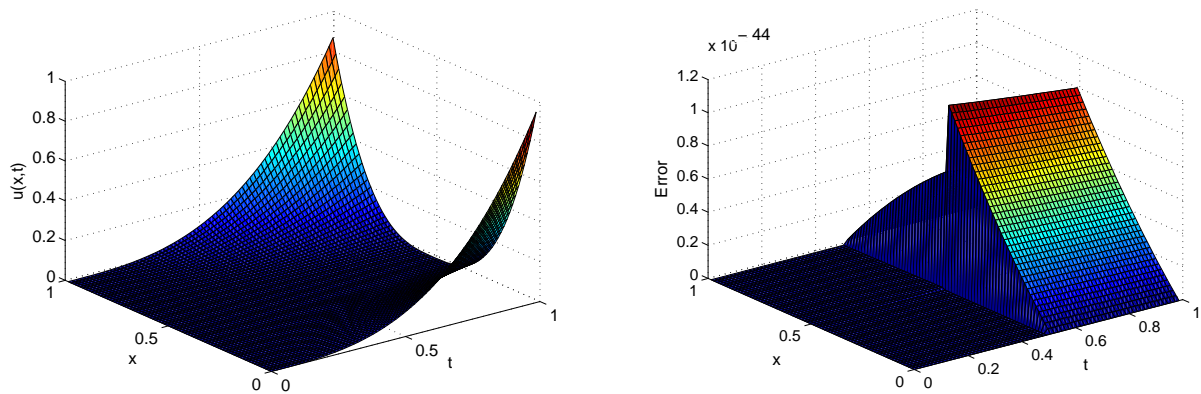


Fig. 4: The numerical behavior of the approximate solution (left) and absolute error (right) for Example 4.

and used to obtain a general approach for forming this matrix. In the proposed method, solution of the problem under consideration was expanded in terms of the the CWs. The operational matrix of variable-order fractional derivative and some properties of CWs were employed to transform its solution to the solution of a linear system of algebraic equations, which greatly simplified the problem as well as achieved a good approximate solution for it. Our proposed method is very efficient and convenient in solving such initial-boundary value problems because all the conditions are used. Also, the implementation of the proposed method is very simple for solution of the problem under consideration. The accuracy of the proposed method was shown for some examples, which shows that our proposed method is very accurate for the problem under study.

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